Towards Robust Local Projections

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Latest Revision: November 14, 2022

Abstract

Local projections with proxy/instrumental variables have been increasingly used for causal inference in empirical macroeconomics. In this study, I propose Bayesian inference for this model under a distributed lagged representation of data. In the proposed parametrization, researchers place priors directly over the structural impulse responses and the first-stage parameter. Dispersed priors over the structural impulse responses allow for weak-instrument robust inference in finite samples and serially correlated data, but this usually leads to poor identification. I show how additional assumptions about the instrument can be incorporated through the first-stage coefficient prior, greatly sharpening inference. To illustrate the method I perform two empirical exercises. First, I estimate US marginal income tax shock's impulse responses on a 16-variable yearly-frequency local projection. Marginal tax shocks are contractionary but their effect on real activity and consumption lasts only two years. I attribute this quick recovery to substitution effects. The second exercise is a replication of Gertler and Karadi (2015) monetary policy analysis using two-year Treasury rates as the policy variable. Even in the presence of a weak instrument, monetary policy shocks are unambiguously contractionary and money is non-neutral after four years.

1 Introduction

Since the seminal work of Sims (1980), structural vector autoregression (SVAR) has been the workhorse model to conduct empirical analysis of macroeconomic fluctuations. Estimation of dynamic responses of outcomes such as output levels, inflation, and unemployment to structural shocks is the primary way macroeconomists conduct *causal* inference. Jordà (2005) proposed Local projections (LP) as an alternative way to estimate these impulse responses (IRs) by projecting leads of outcomes on structural shock time

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†I'm grateful to Pooyan Amir-Ahmadi, Dan Bernhardt, Greg Howard, Shihan Xie and Ji Hyung Lee at the University of Illinois at Urbana-Champaign who provided me support and guidance. I'm also thankful for the comments and discussions many participants provided in the numerous seminars at the University of Illinois at Urbana-Champaign. In particular, I want to thank Collin Philipps, Sebastian Laumer, Vinicios Sant'Anna, Andrea Carolina Atencio De Leon, Cihang Wang, Humberto Martinez-Garcia.

series. This direct approach has been championed by some researchers for providing estimates that are more robust to misspecification than SVARs, at a cost of reduced estimation efficiency (Stock and Watson (2018), Plagborg-Møller and Wolf (2021) and Li et al. (2022)). In particular, LPs are valid even under non-invertibility of data [1]. This is important because many theoretical general equilibrium models with first-order rational expectations imply macroeconomic data is non-invertible, as shown in Sims (2002) and Sims (2012) [2].

In most applications, structural shock observations are not available and a proxy/instrumental variable is needed to identify structural impulse response (Jordà et al. (2015)). Identification, in this case, relies on two assumptions: (1) the instrument is orthogonal to all remaining shocks, at all leads and lags, (2) the instrument is strongly correlated to the structural shock. Naturally, the validity of such assumptions depends both on the instrument and on model specification.

Despite the rising popularity of LPs in the empirical literature, their econometric toolbox is limited to semi-parametric framework. LPs are estimated either via ordinary least squares (OLS) or two-stage least squares (TSLS), and inference is conducted using either Newey and West (1987) residuals or alternative robust residuals such as in Montiel Olea and Plagborg-Møller (2021). Pioneered by Uhlig (2005), Bayesian SVAR has a long tradition in macroeconomics and has benefited from a number of extensions, such stochastic volatility (Uhlig (1997)), factor augmentation (Ahmadi and Uhlig (2015)) or mixed frequency samples (Schorfheide and Song (2015)). Because such extensions require filtering and smoothing, they must be carried out in a full parametric framework and the rich parametrization of such models makes them especially amendable to Bayesian approaches.

In this paper, I derive a full parametric representation of LP regression, with structural IRs identified both directly and with a proxy/instrumental variable. This representation hinges on the assumption the data-generating process is covariance stationary and that fundamental shocks can be recovered from a non-invertible analog of a Gaussian vector moving average (VMA). Such assumptions are compatible with a wide range of possibilities, including linear univariate and multivariate time series models, linear Gaussian state-space extensions of such models, and linear approximations of theoretical dynamic stochastic general equilibrium (DSGE) models. This full parametric representation of LP allows approximate Bayesian inference of structural impulse responses to be well-defined. Using this representation, I propose a Gibb's sampler algorithm to generate draws from the posterior distribution of structural IRs and the nuisance parameters.

Conducting Bayesian inference of LPs (BLP-IV) offers several advantages. First, it eliminates serial corre-

[1] Data is invertible when it can be exactly represented by an autoregression of elements in the sample.

[2] An implication of Plagborg-Møller and Wolf (2021) is that impulse responses are identified in SVAR when using proxy/instruments variable. However, their strategy is only valid for non-weak instruments.

lation in the residuals. Accounting for the serial correlation directly in the model leads to gains in efficiency (See Lusompa (2021) Monte Carlo study) [3]. This helps mitigate LP biggest weakness when compared to alternatives. Second, thanks to shrinkage, Bayesian LPs can be more richly parameterized compared to their classic equivalents. For example, Bayesian LPs can accommodate a much richer set of control variables, even in small samples. Given that instruments/proxy are often correlated with other shocks, control variables play a crucial role in identification. Third, weak instrument concerns can be addressed within the model, as opposed to classic LP which requires alternative weak-instrument robust tests. On Bayesian estimation of IV regressions, the presence of a weak instrument does not invalidate the posterior sampler but it causes the prior distribution to have a predominant influence on the results. Dispersed priors over the impulse responses ensure credible sets adequately reflect the lack of information in weak instrument cases. If the instrument fails to provide enough information for qualitatively meaningful results, the Bayesian LPs can incorporate prior information with respect to the first-stage regression coefficient provided by economic theory. Such restrictions sharpen inference on the presence of weak instruments (Hirano and Porter (2015) and Andrews and Armstrong (2017)) [4]. Additionally, I show that if explicit assumptions about the joint distribution of the structural shock and instruments are made, ranges for the value of the first-stage coefficient can be obtained, giving researchers a necessary condition to validate instruments.

To showcase the method, I perform two empirical exercises. First, I apply the BLP-IV to identify impulse responses of marginal income tax rates in the U.S. economy from 1948-2012. The instrument utilized is constructed from the variation in average marginal tax rates driven by reforms in the tax system or revenue acts. This variation depends on lags of aggregate income. In this case, the exogeneity of the instrument relies on the model's forecasting skill. To address this concern I propose a medium size Bayesian Local Projection with 16 variables and 3 lags. Shrinkage of nuisance parameters allows the richly parameterized model to identify impulse responses in spite of the small yearly sample.

Additionally, under additional theoretical assumptions and exogeneity conditions, I show the parameter of the first-stage regression must be positive and less than one. Using this necessary condition, I assess the quality of instruments commonly used in the literature. By itself, the statutory variation induced by reforms is an endogenous instrument. When interacted with a narrative-driven selection of reforms (Romer and Romer (2009)) the instrument meets the necessary condition.

The results shows marginal income tax shocks are counter-cyclical, with small effects on the impact period that builds up during the next two or three years. The effect on aggregate income is long-lasting

[3] This should be uncontroversial, given that explicit modeling of time series is widespread in forecasting. See Diebold (1998) for a good exposition of the trade-off.

[4] In fact Hirano and Porter (2015) proves without such restrictions an unbiased estimator does not exist for arbitrarily weak instruments.

but fades by the third year in GDP and consumption due to capital-labor displacement effects I'm able to identify.

In the second exercise, I identify monetary policy shocks using a high-frequency instrument as in Gertler and Karadi (2015). When using two-year Treasury rates as the monetary policy variable, the high-frequency instrument is weak. Using dispersed priors over impulse responses combined with the theoretically-based assumption that the correlation between Treasury bond prices and interest rates is positive wields qualitatively meaningful estimates. Monetary policy shocks are counter-cyclical and their effect lasts after 48 months. My estimates also do not present price puzzles or the "one-year puzzle", common in empirical studies of monetary policy shocks.

The paper is outlined as follows: In Section 2 I derive a representation in which the likelihood of LP models can be evaluated. In Section 3 the Gibb's sampler for the posterior distribution of IRs is provided. In Section 4 I provide two empirical applications, first, the identification of marginal tax shocks impulse responses and then the replication of Gertler and Karadi (2015). Finally, Section 5 concludes.

2 The Bayesian Local Projection

I first review local projections and their motivation as a direct approach of estimation and inference over the causal effects of random, unpredictable shocks that drives business cycle fluctuations. As Stock and Watson (2018) I will start by assuming macroeconomic data is represented by a non-invertible analogous of a Vector Moving Average process of infinity order and show, under additional assumptions, local projection residual's likelihood are well defined, and their parameters identify structural impulse responses. Bayesian inference over those impulse responses can then be carried out.

Notation Before presenting the formal econometric results I introduce some notation that is be used throughout this paper. A vector's subscript stands for its period. A superscripts stands for a vector's respective entry e.g. x_t^k is the $k-th$ entry of the vector x at time t. The same applies to matrices. For example $A^{i,j}$ is the (i,j)-th entry of A. As is standard in the time series literature, the subscript $s:t$ aggregates the history between s and t in a $(t+1-s)\times K$ matrix i.e. $x_{s:t}=(x_s', x_{s+1}', ..., x_t')'$. The researcher's set of observations at time t is denoted by $\mathcal{D}_t$ [5]. Identity matrices are denoted by I_i where i is its dimension. The "curly" epsilon ε_t always denotes Gaussian white noise with unit variance, that is $\varepsilon_t \sim^{iid} N(0,1)$.

The technique of writing regression models conditionally on known parameters is extensively throughout the derivation of the sampler. Each transformation will be defined in its own subsection and trans-

[5] Keep in mind this is different from the information set Ω_t, as the latter usually includes non-observables, such as projection residuals and latent states, as well.

formed data is denoted by $\tilde{y}$. Transformations are unique to each subsection.

2.1 Local Projection Estimation of Impulse Responses

Suppose researcher observes a sample of macroeconomic time series $\mathcal{D}_N = \{y_{1:N}\}$, where y_t is a K-dimensional vector. For simplicity, suppose data has been transformed so that a manner that y_t is co-variance stationary and all deterministic factors, including intercepts, trends, and seasonality have been accounted for. These observations are potentially correlated among themselves both contemporaneously and across time. Researchers and policymakers are often interested in understanding the causal impact of economic policy or external events, such as oil supply shortages or military conflicts, in these aggregates. Once one assumes y_t is stationary, representation theorems such as Wold's can be invoked as way to give the time series a certain amount of structure without making explicit assumptions about its distribution. I formalize this idea in the following assumption:

Assumption 1. *Let y_t be time series observations whose joint distribution is covariance stationary. It admits the following representation:*

$$y_t = A_0\epsilon_t + A_1\epsilon_{t-1} + A_2\epsilon_{t-2} + ... \tag{1}$$

where $\{A_l\}_{l=0}^{\infty}$ are $K \times Q$, absolutely summable matrices, $\sum_{l=0}^{\infty} |A_l^{i,j}| < \infty \; \forall \; i = 1,...,K$ and $\forall \; j = 1,...,Q$, and ϵ_t follows Q-dimensional Gaussian white noise with diagonal covariance matrix,

$$\Sigma^{i,j} = \begin{cases} \sigma_i^2 \text{ case } i = j \\ 0 \text{ otherwise.} \end{cases} \tag{2}$$

Furthermore, assume that not all $A_{l \geqslant 1} = 0_{K \times Q}$, to exclude the trivial case where y_t is white noise.

Q can be equal, larger, or smaller than K. The representation is an extension of Wold's theorem to the case where the researcher observes measures of the true, unknown stochastic process, but not the process itself. The appeal of this representation is its broadness, as all linear, stationary state-space models can be represented according to Assumption (2.1) [6]. Such models includes, for example, log-linear approximations of first-order rational expectations DSGE model (Sims (2002)), VARs (when $K = Q$), and dynamic factor models ($K > Q$).

[6] To see this just solve the measurement equation by backwardly substituting the state-transition. Recall I'm neglecting deterministic and non-dynamic terms in this representation.

One of the original motivations of local projections, as proposed by Jordà (2005), is consistency and unbiasedness of impulse response estimates when the VMA representation is non-invertible ($Q \neq K$). On an intuitive level, this allows researchers to contemplate structural shocks without a one-to-one mapping between them and observed time series. For example, consider income tax shocks (my first application). Income taxation has several dozen possible measures: average and marginal income tax rates at different income levels, size of the tax base, deduction possibilities, and taxation of different income categories such as wages and capital gains, etc. All these measures could be part of the set of observations $\mathcal{D}$. The same legislative change may perturb several of those measures simultaneously, precluding the plausibility of orthogonal structural shocks for each measure. Defining tax shocks simply as policy changes that are orthogonal to the remaining structural shocks as in Romer and Romer (2010) is more convincing. Tax shocks having a different dimension than policy measures do not present a conceptual challenge to the proposed representation (2.1).

Let ϵ_t^q represent the particular policy or event the researcher is interested in, such as marginal income tax shocks and monetary policy shocks as in the case of this paper. Its causal impact on aggregate outcome k, h periods after its arrival, is commonly defined as the impulse response (3), a simple expected counterfactual variation. The objective of local projection analysis is to estimate and conduct inference over those impulse responses:

$$\beta_h^k \equiv E[y_{t+h}^k|\epsilon_t^q = 1] - E[y_{t+h}^k|\epsilon_t^q = 0]. \tag{3}$$

Now, consider the linear projection of some outcome k over the shock of interest:

$$y_{t+h}^k = \rho\epsilon_t^q + u_{t+h}^k \text{ such that } (y_{t+h}^k - \beta_{k,h}\epsilon_t^q) \perp \epsilon_t^q. \tag{4}$$

According to (1), the projection error u_{t+h}^k is a linear combination of all shocks from $t+h$ to the infinity past except for ϵ_t^q, $u_{t+h}^k = \sum_{l=0}^{\infty} \alpha_l^{k'} \epsilon_{t+h}$. The coefficient vector α_l^k is the k-th line of matrix A_l, except for α_h^k whose q-ith entry is exactly zero. By taking conditional expectations over (4), one can verify ρ identifies the impulse response β_h^k.

Surprisingly even though a general dynamic model was assumed, *mean-identification* of impulse response does not require an explicit estimation of transmission channels between macroeconomic time series or consideration over its dynamic properties. The OLS estimator of y_{t+h}^k over ϵ_t^q is an unbiased and consistent estimator of the impulse responses. In practice, such a regression is not feasible unless observations of the structural shock of interest ϵ_t^q are available. Identification of the impulse responses must be carried out with an external instrument z_t [7]. In section (2.1.2) I will detail how to carry out this form of identification. For now, I will discuss a second issue.

In addition to the fact ϵ_t^q is generally not observable, Bayesian inference over β_h^k is infeasible, as it is, for a second reason. The projection residuals u_{t+h}^k are serially correlated through the common shock terms $\epsilon_{t+h}^{k \neq q}$, making estimation of their covariance non-trivial. To see this, consider their autocovariance function $s = 0, 1, 2, ...$

$$\gamma^k(s) = \sum_{l=0}^{\infty} \sum_{r=0}^{\infty} \alpha_l^{k\prime} \Sigma^* \alpha_r^k \tag{5}$$

$$\Sigma^* = \begin{cases} \Sigma & \text{if } l = r + s \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Note from (5) that: (i) u_{t+h}^k is autocorrelated and of an arbitrarily high order; (ii) the right-hand-side does not depend on t, so that e_{t+h}^k is covariance stationary. Those two observations imply $E(u^k u^{k\prime})$ is a symmetric Toeplitz matrix with unique serial correlation terms equal to the sample size $N - h$. Even if one could derive a posterior sampler for such a matrix, a daunting task in itself, this posterior wouldn't have desirable large sample properties.

For those reasons, evaluating the likelihood through (4) is not practical. However, there is a simpler alternative: evaluate errors of forecasting models of y_t. As I show in the next section, those forecasting errors have a known covariance structure (which is exploited to construct a filter) and the projection of ϵ_t^q over those forecasting errors identify the impulse responses. This is the parametric equivalent of the semi-parametric LPs. The trade-off between the parametric and semi-parametric methods is the standard one - parametric models are more efficient when correctly specified, but less robust to distributional assumptions.

2.1.1 Representation when ϵ_t^q is known.

In this section assume the researcher observe the shock of interest ϵ_t^p in addition to the macroeconomic time series, $\mathcal{D}_N = \{y_{1:N}, \epsilon_{1:N}^q\}$. The goal is to conduct Bayesian inference over the impulse response, β_h^k. Although this assumption is quite strong in general, this exercise is useful for multiple reasons. First, it showcases the issues with LPs parametric representation that are tangential to the identification of β_h^k. Second, this is a necessary step in the derivation for the case of ϵ_t^q is identified with an instrument z_t. Third, there are

[7] Alternatively, one can impose restrictions on the matrices A_l as Plagborg-Møller (2019). However, this paper focuses on the former

cases where a first stage is not necessary, for example, when using a shock . In those cases, the following posterior sampler could be used to simulate posteriors of impulse responses.

First of all, let $\Omega_t = \{\epsilon_s\}_{s=-\infty}^{t}$ be an information set at time t which spans the space of ϵ_t [8]. Let the optimal linear h-step ahead forecasts of each individual time series be given by $y^k_{t+h,t-1} = \text{Proj}(y^k_{t+h}|\Omega_{t-1})$ and its forecast error $u^k_{t+h,t-1} = y^k_t - y^k_{t+h,t-1}$. The following theorem yields a parametrization of local projections that can be used in maximum likelihood and Bayesian methods:

Theorem 1. *Let y_t be a stochastic process that follows (1) and let ϵ^q_t be the structural shock series of interest. The parameter of the projection of forecasting errors be $u^k_{t+h,t-1}$ over ϵ^q_t identify the h-step ahead impulse response. That is, let ψ be such that*

$$u^k_{t+h,t-1} = \psi\epsilon^q_t + v^k_{t+h,t-1} \text{ such that } \epsilon^q_t \perp v^k_{t+h,t-1} \tag{7}$$

then, $\psi = E(y^k_{t+h}|\epsilon^q_t = 1) - E(y^k_{t+h}|\epsilon^q_t = 0)$.

Proof. By construction $\psi = E(u^k_{t+1,t-1}|\epsilon^q_t = 1) - E(u^k_{t+1,t-1}|\epsilon^q_t = 0)$. One only needs to show $E(u^k_{t+h,t-1}|\epsilon^q_t) = E(y^k_{t+h}|\epsilon^q_t)$. In the Gaussian case, $\text{Proj}(y_{t+h}|\Omega_{t-1}) = E(y^k_{t+h,t-1})$, which, from (1), is a linear combination of shocks ϵ_t within the information set Ω_{t-1}. Since $\epsilon^q_t \notin \Omega_{t-1}$, $\epsilon^q_t \perp y_{t+h,t-1}$. As result, $E(y^k_{t+h,t-1}|\epsilon^q_t) = E(y_{t+h,t-1}) = 0$. Taking expectations conditional on ϵ^q_t over $y_{t+h} = y_{t+h,t-1} + u_{t+h,t-1}$ concludes the proof. □

At first glance, the parametrization (7) may not seem very useful, as the projection error $v^k_{t+h,t-1}$ is still serially correlated for $h \geqslant 1$. However, the following result establishes $v^k_{t+h,t-1}$ has MA(h) representation hence usual techniques used in likelihood-based and Bayesian analysis of ARMA process can be used.

Corollary 2. *Define $v^k_{t+h,t-1}$ as in Theorem 1. There exist $\phi_{1:h} \neq 0$ such that:*

$$v^k_{t+h,t-1} = e^k_{t+h} + \phi_1 e^k_{t+h-1} + ... + \phi_h e^k_t \tag{8}$$

$$e^k_{t+h} \sim \mathcal{N}(0, \sigma^2_k) \tag{9}$$

Proof. See Appendix. □

Theorem (1) and Corollary (2) together motivate the parametric representation of LPs that I use for the remainder of this work. They imply that if research includes enough controls such that they are able to

[8] This is not to be confused with observation set $\mathcal{D}_t$.

predict y_t^k to a sufficient degree, the residuals of the LP regression will have exact MA(h) representation. Henceforth, I consider LPs of the form:

$$y_{t+h}^k = \beta_h^k \epsilon_t^q + \gamma^k w_{t-1} + e_{t+h}^k + \phi_1 e_{t+h-1}^k + ... + \phi_h e_t^k, \tag{10}$$

where w_t is a vector of controls rich enough such that the forecasts $\gamma^k w_{t-1}$ approximate the population projection of the target outcome variable. To formalize this notion, I make it explicit as additional assumption:

Assumption 2. *The vector of controls w_t is such that $\gamma^{k\prime} w_t \approx Proj(y_{t+h}|\Omega_{t-1})$.*

About the representation (10), several remarks are in order:

1. It is valid even when $Q \neq K$ (y_t is non-invertible). As I show in the appendix, $u_{t+h,t-1}$ has a vector moving average (VMA) representation, because it is covariance stationary. However, one can verify e_{t+h} is *not* a linear transformation of ϵ_t. The structural shocks cannot be retrieved from e_{t+h}.

2. Note that a univariate parametrization for $v_{t+h,t-1}^k$ can be achieved though y_t is multivariate. This is indicative that cross-correlations between k and $j \neq k$, as well their autocorrelations, are nuisance parameters, as first argued by Jordà (2005).

3. The fact that $\gamma^k w_{t-1}$ needs to approximate Proj$(y_{t+h}|\Omega_{t-1})$ in order for the parametrization (10) to be valid motivates my emphasis on usage of models with good forecasting performance, such as medium to large size time series models, models with time-varying parameters, dynamic factor models, etc.

4. The term $\gamma^k w_{t-1}$ is fairly flexible and one can interpret it as the measurement of a linear state-space model. It follows that all the approaches cited above can be easily implemented by taking (10) as the measurement equation.

5. Although forecasting and causal inference are fundamentally different objectives, the former is a prerequisite for the latter when using such parametrization of LPs. Researchers using (10) should not dismiss a model's forecasting performance. While this consideration may seem important only in the likelihood and Bayesian contexts, it is well-known that an equivalent condition is required to relax

the (often strong) lag-exogeneity condition of the instrument.

6. Under the Gaussian distributional assumption, the required approximation becomes $\gamma^k w_t \approx E(y_{t+h}|\Omega_{t-1})$. In that case, the validity of the parametrization only relies on *accuracy* of forecasts, not their *precision*. For this reason, traditional model selection criteria, which typically minimize RMSE, cannot be used in this context. For the purpose of identifying impulse response functions with LPs, all emphasis should be put on forecast bias reduction, which suits LPs.

Inference of (10) can be done using any method that allows for moving-averaging residuals. There are several such methods, including textbook applications of Kalman smoothing as in Hamilton (2020). In the Bayesian case with conforming priors, Chib and Greenberg (1994) algorithm can be applied without any modifications. We next consider the case where ϵ_t^q is unobserved but an external instrument z_t for it is available.

2.1.2 Representation with Instruments

The procedure described above is infeasible as the shock of interest ϵ_t^q cannot be observed. However, many structural shocks can be traced back to concrete policy changes or events, and information about such events is available to researchers. In my marginal income tax setting, legislative records and congressional reports provide ample information about changes to tax policy and their motivation. In the case of monetary policy, researchers exploit the fact that policy changes to interest rates must be first announced by the Federal Open Market Committee (FOMC). In both cases, the external information provides the dates on which shocks *do not* arrive and a dummy variable can be constructed to capture this movement. In practice, proxies are constructed to capture as much of the assumed exogenous variation as possible using both narrative accounts and "external data", data that are not included as controls w_t.

Previously, some researchers used to assume those constructed shocks stood as perfect proxies of structural shocks, such as in Romer and Romer (1989). The algorithm described in the last session could be used to implement such an approach by simply assuming $z_t = \epsilon_t^q$. While economic theory can provide good justification for why a proxy approximates the desired structural shock, it cannot ensure the proxy captures the full extent of the structural shock variation, or do so without measurement errors. For this reason, the assumption $z_t = \epsilon_t^q$ is too strong. Since Mertens and Ravn (2012), many researchers have instead treated the proxies constructed from narrative accounts and external data as instrumental variables / external instruments.

Let be z_t be the instrumental variable. Consider the coefficient of $\tilde{y}_{t+h}^k = y_{t+h}^k - \gamma' w_t$ over z_t:

$$\delta_h^k = \frac{E(\tilde{y}_{t+h}^k z_t)}{E(z_t^2)} \quad (11)$$

$$\delta_h^k = \beta_h^k \frac{E(\epsilon_t^q z_t)}{E(z_t^2)} + \frac{E(v_{t+k,t-1}^k z_t)}{E(z_t^2)}. \quad (12)$$

The coefficient δ_h^k does not identify β_h^k without additional identifying assumptions. The first set of identifying assumptions relates to the second term on the right-hand-side:

Assumption 3. *There is a instrument $z_t \in \mathcal{D}$ that it is lead and contemporaneously exogenous:*

1. $E(z_t \epsilon_t^{p \neq q}) = 0$
2. $E(z_t \epsilon_{t+1:h}) = 0$.

Thus, $E(u_{t+k \perp t}^k z_t) = 0$ *for every* $k = 1, ..., K$.

Assumption 3 are the exogeneity conditions for instrumental variable regressions in the context of local projections, though the lag-exogeneity condition is unnecessary. Both conditions must be met through the construction of the instrument. Under Assumption 3 δ_h^k simplifies to:

$$\delta_h^k = \beta_h^k \pi, \quad (13)$$

where $\pi = \frac{E(\epsilon_t^q z_t)}{E(z_t^2)}$. That is, the projection coefficient of a suitable instrument over $y_{t+h \perp t}^k$ identifies the product of the desired impulse response times the projection coefficient of the instrument over the unobserved structural shock. Substituting a suitable instrument z_t for the shock in the algorithm described in subsection 2.1.1 identifies β_h^k, but with an unknown scale. Bayesian inference of structural impulse responses requires joint inference of both β_h^k and π. To do so, I normalize the impulse responses as follows:

Assumption 4. *Let y_t^1 be the first variable in the vector y_t. Its impulse response at impact-period is normalized to one, $\beta_0^1 = 1$, such that*

$$y_t^1 = \pi z_t + \gamma_0^{1\prime} w_t + e_t^1. \quad (14)$$

This is a real assumption and not just normalization because it requires $\beta_0^1 \neq 0$. That is, we are assuming ϵ_t^q impulse responses are not zero for the policy variable. Note normalization does *not* need to be done with respect to y_t^q. For example, suppose a small trivariate VAR of government revenues ($k = 1$), government spending ($k = 2$), and some measure of real activity ($k = 3$). We could potentially normalize β_0^1 even if when using an instrument for spending. In that case, we interpret impulse responses as changes caused by spending shocks that raise revenues by one unit. This makes our normalization scheme exactly the same as that for proxy/instrumental variables VARs. One practical consequence of this feature is that one can draw inferences with respect to ϵ_t^q even when y_t^q is not available.

To perform Bayesian inference over β_h^k using an instrument, one needs to estimate π as well. Evaluating equation (14) at $t+h$ yields:

$$y_t^1 = \pi z_t + \gamma_0^{1\prime} w_t + e_t^1 \tag{15}$$

$$y_{t+h}^k = \beta_h^k \pi z_t + \gamma_h^{k\prime} w_t + e_{t+h}^k + \phi_1^k e_{t+h-1}^k + ... + \phi_h^k e_t^k \tag{16}$$

$$\begin{pmatrix} e_{t+h}^1 \\ e_{t+h}^k \end{pmatrix} \sim iidN(\mathbf{0}, \Sigma_h^k) \tag{17}$$

Equation (15) is the first stage. Equation (16) is the second stage. Together with the distributional assumption (17), they form the empirical model.

First, notice, that due the exogeneity conditions, π is the only common parameter across both linear projections. This local projection can be parametrized as a restricted reduced form instrumental variable regression with unusual dynamic structure. This parameterization, as opposed to estimating δ_h^k and dividing by π, allows to implement priors that are approachable for weak instrument inference: (i) the model directly identifies the impulse response β_h^k, allowing dispersed priors over those; (ii) the restricted reduced form also avoids the "divide by zero" problem that arises when the instrument is not valid, $\pi = 0$, or close to zero [9].

Second, notice the correlation between e_{t+h}^1 and e_{t+h}^k is *not* what gives rise to endogeneity in our model, in contrast with traditional formulations of two-stage regressions. That is, even if y_t^1 and y_t^k have no shocks in common other than the structural shock and Σ_h^k is a diagonal matrix, the projection coefficient of y_t^k on y_t^1 wouldn't identify β_h^k unless $e_t^1 = 0$ and ϵ_t^q is the only structural shock responsible for y_t^1 variation. This identification scheme would be akin to recursive identification.

Finally, I do not assume $\pi \neq 0$ *a priori*. A weak instrument does not contradict the empirical model (15)-(17), although it reduces the identifying variation used to update the posterior of β_h^k.

[9] See Kleibergen and Zivot (2003) for details of alternative Bayesian parametrizations of IV regressions. Notice my priors are differ from theirs Bayesian two-stages, although both models are in restricted reduced form.

3 Bayesian Inference

In this section I describe the posterior sampler of equations (15)-(17) parameters. The residuals with moving average dynamics $e_t^k, ..., e_{t+h-1}^k$ will be treated as latent states, s_{t+h}. Let $\phi = (\phi_1, ..., \phi_h)'$, $s_{t+h} = (e_t^k, ..., e_{t+h-1}^k)'$, $S = s_{T:(T+h)}$, $\gamma = (\gamma_0^1, \gamma_h^k)'$. To simplify notation I will also omit k and h from β and Σ. The target posterior distribution is given by $p(\pi, \beta, \gamma, \phi, \Sigma, S|\mathcal{D})$.

The moving average dynamics and the non-linearity of the RRF makes it difficult to derive an expression of (15)-(17) likelihood and a direct sampler for the posterior distribution. I instead use Gibb's sampler. The partitioning of the parameter space and the respective conditional posteriors are

$$S|\pi, \beta, \gamma, \phi, \Sigma, \mathcal{D} \tag{18}$$

$$\Sigma|\pi, \beta, \gamma, \phi, S, \mathcal{D} \tag{19}$$

$$\phi|\pi, \beta, \gamma, \Sigma, S, \mathcal{D} \tag{20}$$

$$\beta, \gamma|\pi, \phi, \Sigma, S, \mathcal{D} \tag{21}$$

$$\pi|\beta, \gamma, \phi, \Sigma, S, \mathcal{D}. \tag{22}$$

Now I proceed to derive each conditional posterior distribution and as briefly discuss the choice of prior distributions.

3.0.1 Step 1: $p(S|\pi, \beta, \gamma, \phi, \Sigma, \mathcal{D})$

Given the conditionals, evaluate moving average residuals $u_{t+1}^1 = e_{t+h}^1$ and $u_{t+h}^k = e_{t+h}^k + \phi_1 e_{t+h-1}^k + ... + \phi_h e_t^k$. They form a bivariate restricted VMA(h) model where one of the series is white noise. Kalman smoother provides draws from the posterior $p(S|\pi, \beta, \gamma, \phi, \Sigma, \mathcal{D})$. Assume a dispersed, conforming prior for initial conditions, $S_0 \sim \mathcal{N}_h(0, I_h {a_0^S}^2)$, where ${a_0^S}^2 \to \infty$. See Durbin and Koopman (2012) for details.

3.0.2 Step 2: $p(\Sigma|\pi, \beta, \gamma, \phi, S, \mathcal{D})$

Conditional on the state variables, the iid reduced form shocks can be evaluated:

$$e_{t+h}^1 = y_{t+h}^1 - \pi z_{t+h} - \gamma_0^1 w_{t+h} \tag{23}$$

$$e_{t+h}^k = y_{t+h}^k - \pi z_{t+h} - \gamma_h^k w_t - \rho_1 e_{t+h-1}^k - ... - \rho_h e_t^k \tag{24}$$

Under the assumption of Gaussian errors and with Inverse Wishart priors, the conditional posteriors has closed form solution:

$$p(\Sigma) \sim \mathcal{IW}(V_0, N_0) \tag{25}$$

$$p(\Sigma|\pi,\beta,\gamma,\phi,S,\mathcal{D}) \sim \mathcal{IW}(V_0 + e'e, N_0 + T - h), \tag{26}$$

where $e = (e^1_{h:(t+h)}, e^k_{h:(t+h)})'$.

3.0.3 Step 3: $p(\phi|\pi,\beta,\gamma,S,\Sigma,\mathcal{D})$

Under this conditioning set, evaluate the model as

$$y^1_{t+h} - \pi z_{t+h} - \gamma^1_0 w_{t+h} = e^1_{t+h} \tag{27}$$

$$y^k_{t+h} - \pi\beta^k_h z_t - \gamma^k_h w_t = \phi s_{t+h} + e^k_{t+h} \tag{28}$$

Since e^1_{t+h} is conditionally observed through equation (27), I rewrite the model in a single equation conditional on e^1_{t+h} realization:

$$y^k_{t+h} - \pi\beta^k_h z_t - \gamma^k_h w_t = \phi s_{t+h} + e^k_{t+h}|e^1_{t+h}, \tag{29}$$

where $e^k_{t+h}|e^1_{t+h} \sim \mathcal{N}(\mu_{k|1}, \sigma^2_{k|1})$, $\mu_{k|1} = \Sigma^{1,2}\Sigma^{2,2^{-1}}$ and $\Sigma^{k|1} = \Sigma^{1,1} - \Sigma^{1,2}\Sigma^{2,2^{-1}}$. Standardizing equation (29) yields

$$(y^k_{t+h} - \pi\beta^k_h z_t - \gamma^k_h w_t - \mu_{k|1})\sigma^{-1}_{k|1} = \phi(s_{t+h}\sigma^{-1}_{k|1}) + \varepsilon \tag{30}$$

$$\tilde{y}_{t+h} = \phi\tilde{s}_{t+h} + \varepsilon. \tag{31}$$

Equation (31) is Bayesian linear regression with unit variance. Since I explicitly assumed non-invertibility, ϕ is not uniquely identified in the entire parameter space. To avoid computational problems, the support needs to be truncated to invertible region with an indicator prior distribution (32), as suggested by Chib and Greenberg (1994).

$$p(\phi) \sim N(\phi_0, A_0^{-1}) \mathbb{1}(\phi \in R_\phi) \tag{32}$$

$$p(\phi|\pi, \beta, \gamma\Sigma, S, \mathcal{D}) \propto p(\tilde{y}_{1:T+h}|\phi, \pi, \beta, \gamma\Sigma, S, \mathcal{D} - y_{1:T+h}) N(\phi_0, A_\phi^{-1}) \mathbb{1}(\phi \in R_\phi), \tag{33}$$

where $\mathbb{1}$ stands for an indicator function and R_ϕ is the invertible region. The resulting posterior distribution given by (33) is non-standard and draws must be generated using the Metropolis-Hasting algorithm. I use the posterior of regression coefficients of (31) when $\phi \in R_\phi$ as the proposal density, $N((\tilde{S}'\tilde{S} + A_\phi)^{-1}\tilde{S}'\tilde{y}, (\tilde{S}'\tilde{S} + A_\phi\phi_0)^{-1})\mathbb{1}(\phi \in R_\phi)$.

3.0.4 Step 4: $p(\beta, \gamma|\pi, \phi, \beta, \Sigma, S, \mathcal{D})$.

First adjust the model to the conditional set:

$$y_{t+h}^1 - \pi z_t = \gamma_0^1 w_{t+h} + e_{t+h}^1 \tag{34}$$

$$y_{t+h}^k - \phi s_{t+h} = \beta_h^k(\pi z_t) + \gamma_h^k w_t + e_{t+h}^k, \tag{35}$$

equation (34) is simply a system of seemingly unrelated regressions with known covariance Σ_h^k. Such models have conforming, Gaussian priors given by equation (36). The posterior immediately follows from the model standardized in matrix form

$$p(\beta, \gamma) \sim N(\mu_0, A_\mu^{-1}\mu_0) \tag{36}$$

$$p(\beta, \gamma|\pi, \phi, \beta, \Sigma, S, \mathcal{D}) \sim N((\tilde{X}'\tilde{X} + A_\mu)^{-1}(\tilde{X}'\tilde{y} + A_\mu\mu_0)), \tag{37}$$

where

$$D = (\Sigma_h^k \otimes I_{T-h})^{-1/2} \tag{38}$$

$$\tilde{y} = D \begin{pmatrix} y_{1:T+h}^1 \\ y_{1:T+h}^k \end{pmatrix} \tag{39}$$

$$\tilde{X} = D \begin{pmatrix} w_{1:T+h} & 0 & 0 \\ 0 & w_{1:T} & \pi z_{1:T}, \end{pmatrix} \tag{40}$$

3.1 Step 5: $p(\pi|\beta, \phi, \gamma\Sigma, S, \mathcal{D})$

For draws of the first stage parameter π, one can use a similar device as in step 4. However the system of equations is not seemingly unrelated since both regressions, by assumption, share the same parameter.

$$y^1_{t+h} - \gamma^1_0 w_{t+h} = \pi z_{t+h} + e^1_{t+h} \tag{41}$$

$$y^k_{t+h} - \gamma^h_h w_t - \phi s_{t+h} = \pi(\beta^k_h z_t) + e^k_{t+h}. \tag{42}$$

The system formed by equations (41) and (42) can be rewritten as single linear regression by stacking z_{t+h} and $\beta^k_h z_t$:

$$\tilde{y} = \tilde{Z}\pi + \tilde{e}, \tag{43}$$

where,

$$D = (\Sigma^k_h \otimes I_{T-h})^{-1/2} \tag{44}$$

$$\tilde{y} = D \begin{pmatrix} y^1_{1:T+h} - \gamma^1_0 w_{t+h} \\ y^k_{1:T+h} - \gamma^h_h w_t - \phi s_{t+h} \end{pmatrix} \tag{45}$$

$$\tilde{Z} = D \begin{pmatrix} z_{1:T+h} \\ \beta^k_h z_{1:T}. \end{pmatrix} \tag{46}$$

The simplicity of equation (43) is what gives almost total flexibility in the choice of prior distribution $p(\pi)$ as it is always straightforward to derive a sampler with non-conforming priors for univariate, linear regressions. For example, priors with bounded support can be used to impose sign restrictions on this first-stage parameter. One of the central arguments of this paper is that economic theory that justifies instrument validity can also be used to obtain likely ranges for π. For this reason I leave π prior unspecified, since it is ultimately depends on the context of the application. Next I discuss some common cases.

3.2 Connecting Instrument Designs to First Stage Parameters π

Consider the example of narrative identification with nothing but sign indicators:

$$z_t = \begin{cases} -1 \text{ if } g(\epsilon_t^q + m_t) < \underline{a} \\ 0 \text{ if } \underline{a} \leqslant g(\epsilon_t^q + m_t) \leqslant \bar{a} \\ 1 \text{ if } g(\epsilon_t^q + m_t) > \bar{a}, \end{cases} \tag{47}$$

where the constructed external instrument z_t is meant to identify the shock ϵ_t. Similar examples have been proposed by Plagborg-Møller and Wolf (2021) and Boer and Lütkepohl (2021). Sufficiently large shocks are connected to events recorded by the researcher, such as oil supply disruptions, war-time spending or tax reforms. If the function $g()$ is increasing, researchers guess the sign of the shocks correctly, on average. The function $g()$ and the bands $(\underline{a}, \bar{a})$ determine how often shocks are recorded, as well as possible skewness in the recording [10]. Finally, the recording might be contaminated by measurement error m_t. Under (47) it can be shown the first stage parameter is

$$\pi = \frac{E(\epsilon_t | g(\epsilon_t^q + m_t) > \bar{a}) - E(\epsilon_t^q | g(\epsilon_t^q + m_t) < \underline{a})}{p(g(\epsilon_t^q + m_t) < \underline{a}) + p(g(\epsilon_t^q + m_t) > \bar{a}) - (p(g(\epsilon_t^q + m_t) < \underline{a}) - p(g(\epsilon_t^q + m_t) > \bar{a}))^2} \geqslant 0, \tag{48}$$

where $p()$ is the joint probability density function of (ϵ_t^q, m_t). It is positive than zero since both probabilities in the denominator are between 0 and 1, while the numerator is strictly positive because $E(\epsilon_t^q | g(\epsilon_t^q + m_t) \geqslant \bar{a}) > E(\epsilon_t^q | g(\epsilon_t^q + m_t) \leqslant \underline{\text{a}})$. Instrument weakness in this context can be characterized by $(\underline{a}, \bar{a})$ being so wide that few shocks are recorded or measurement error being so high that $E(\epsilon_t^q | g(\epsilon_t^q + m_t) \geqslant \bar{a}) \approx E(\epsilon_t^q | g(\epsilon_t^q + m_t) \leqslant \underline{\text{a}})$.

It is also possible to find an upper bound for π. The highest performance sign-based external instrument for ϵ_t^q is when its sign is always recorded and done so correctly. In other words, $\underline{\text{a}} = \bar{a} = 0$ and $m_t = 0$. In that case, the first stage parameter simplifies to

$$\pi = E|\epsilon_t^q|,$$

which is strictly positive, finite under mild assumptions and proportional to ϵ_t in scale. For example, if structural shock is Gaussian, $|\epsilon_t^q|$ is half-normal distributed and $\pi \propto \sigma$. Unfortunately, because $E|\epsilon_t^q|$ is not observable, it's not possible to impose any more restrictions on π parameter space from (47) alone. This is a direct consequence of using instruments that are not of the same scale as ϵ_t^q.

[10] For example, oil supply shocks would be mostly negative, so that $\bar{a} \to \infty$.

Recall the unit-scale normalization fixes the scale of the structural shock ϵ_t^q as the scale of y_t^1. In many applications, instruments can be constructed to be in that same scale as well. In that case the instrument design is given by:

$$z_t = \begin{cases} 0 \text{ if } \underline{a} \leqslant g(\epsilon_t^q + m_t) \leqslant \bar{a} \\ e_t + m_t \text{ otherwise} . \end{cases} \tag{49}$$

That is, for small shock realizations, no narrative is recorded. When large shocks realize, the instrument is constructed to capture its full effect, but can still be contaminated by measurement error m_t. If instruments are given by (49) the first stage parameter is

$$\pi = \frac{E({\epsilon_t^q}^2|G(\epsilon_t^q, m_t)) + E(\epsilon_t^q m_t|G(\epsilon_t^q, m_t))}{E({\epsilon_t}^2|G(\epsilon_t, m_t)) + 2E(\epsilon_t m_t|G(\epsilon_t, m_t)) + E(m_t^2|G(\epsilon_t, m_t))}, \tag{50}$$

where $G(\epsilon_t, m_t)) = \{g(\epsilon_t, m_t) \notin [\underline{a}, \bar{a}]\}$. Since instrument is more likely to record when ϵ_t^q and m_t have the same sign, $E(\epsilon_t^q m_t|G(\epsilon_t^q, m_t)) > 0$, henceforth $\pi < 1$.

In all these examples, the sign of π is derived from the theoretical assumptions. Any choice of prior distribution with support in the real line incorporates this information. Uniform priors in particular are convenient as the posterior distribution of π has a closed-form solution

$$p(\pi) \sim U(0, \pi_0) \tag{51}$$

$$p(\pi|\beta, \gamma, \phi, \Sigma, \mathcal{D}) \sim N_{[0,\pi_0]}(\tilde{Z}'\tilde{Z})^{-1}\tilde{Z}'\tilde{y}, \tilde{e}'\tilde{e}), \tag{52}$$

where $N[0, \pi_0](\mu, \sigma^2)$ is the Gaussian distribution truncated at $[0, \pi_0]$.

4 Empirical Applications

To illustrate the Bayesian LP-IV, I use it to estimate impulse responses of economic policy in two different applications. First application I identify marginal income tax shocks in US economy, as in Mertens and Ravn (2012) and Mertens and Montiel Olea (2018). Second, I estimate the effect of monetary policy shocks, also in the United States economy, as in Gertler and Karadi (2015) using the BLP-IV.

4.1 Marginal Income Tax Shocks

A central question in public policy is, to what extent marginal income tax rates matters for individual decisions to work, invest and innovate? Governments that find themselves with high debt and slow growth often raise taxes to boost tax revenue, even though raising taxes could hurt activity in the short run. Income tax cuts are also a common counter-cyclical policy, such as the American Tax Cuts and Jobs Acts of 2017.

The empirical literature studying US individual tax returns [11] finds that changes in marginal income only have a modest impact on aggregate income. This is puzzling, as the empirical macro literature often finds that average marginal tax rates (AMTR) are an important factor in explaining fluctuations in economic activity and unemployment. Mertens and Montiel Olea (2018) made a significant contribution by bringing individual tax return data into dynamic macro models and their evidence largely supports the findings of macroeconomics literature — marginal tax shocks do affect activity.

In this application, I extend the analysis of AMTR shocks in several ways. To better explain the contributions of the Bayesian LP-IV, let me first introduce the instruments used.

4.1.1 AMTR Instrument

The instrumental variable/external instrument z_t used to capture exogenous variation in income tax policy is given by

$$z_t = d_t \times \frac{1}{M_t} \sum_{i=1}^{M_t} (\tau_t(\text{income}_{t-1,i}) - \tau_{t-1}(\text{income}_{t-1,i})), \tag{53}$$

where $income_{t-1,i}$ is payroll taxes plus individual taxable income, defined as all sources of income excluding capital gains and government transfers.[12] The function τ_t income tax schedule at period t and d_t is a dummy variable. This function maps the declared individual taxable income to its marginal income tax during tax year t. The quantity $(\text{income}_{t-1,i}) - \tau_{t-1}(\text{income}_{t-1,i}))$ measures the marginal income tax variation caused by only changes in the schedule between periods $t-1$ and t. This quantity was measured for a sample of M_t individuals. The total average of those measures is the statutory variation (SV) in tax policy which captures mechanical variation in AMTR during policy changes, discounting the effects of fluctuations in the tax base. These measures are provided in Mertens and Montiel Olea (2018) replication files.

Using the SV as instrument faces two potential identification problems. First, some tax reforms may

[11] See Saez et al. (2012) for a survey.

[12] This is the definition most commonly used in the taxable income literature. See Piketty and Saez (2003).

not work as an appropriate basis to construct tax shocks if the reforms are anticipated or are done in response to contemporaneous events. The literature has long used Romer and Romer (2009) narrative of US tax reforms as guidance to select suitable reforms, which is done through the d_t interaction term.[13] Unfortunately, a consensus regarding which reforms should be included is hard to come by and as I show, results are sensitive to this choice. For this reason, I run the models with three different instrumental variables:[14]

- IV1: Includes all statutory variations. That is, $d_t = 1$ for all sample.
- IV2: Includes all tax shocks classified by Romer and Romer (2009) as exogenous changes.
- IV3: A subset of IV2, include only reforms classified as exogenous by Mertens and Montiel Olea (2018).

A second identification issue comes from instruments being constructed from lagged taxable income which is expected to be strongly correlated with other lagged shocks. Without a suitable set of controls and an adequate number of distributed lags, the instrument violates the lag-exogeneity condition. A rich set of controls is a challenge in this context, as income taxation has a yearly frequency, resulting in a relatively small sample ($N = 65$). It is also well known in the micro panel literature that just controlling for lags of income does not correct these biases. [15] The reason is that several macro level shocks that are correlated with the regression outcome y^k_{t+h} may only affect aggregate income with a delay. For example, the effect of productivity shocks on aggregate income tend to "build over time". Since the contemporaneous correlation between the two is not strong, lags of taxable income do not account for this. To address this concern I use a relatively large set of controls together with the shrinkage priors described in the previous session. An additional advantage of this larger model approach is that we are also able to identify a broader range of effects.

4.1.2 Instrument Validation

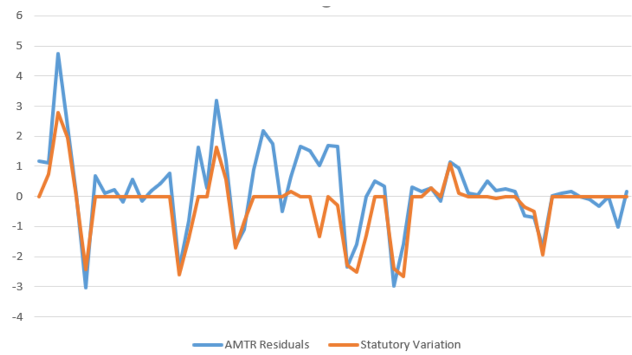
To showcase the heuristic presented in section 3.2, I propose an explicit model for (z_t, ϵ^q_t). Consider again how the instrument is constructed: whenever the narrative, modeled through the dummy d_t, identifies a revenue act or episode of tax reform, the external instrument is exactly equal to the statutory variation. In that case, differences between ϵ^q_t and z_t must come from either:

[13] They attribute four possible justifications to each tax reform: (a) response to current or future adjustments to government spending; (b) offsetting cyclical fluctuations; (c) addressing long term debt growth; (d) stimulate investment and/or long-term growth. The literature considers the last two as candidates for tax shocks.

[14] See the appendix for a table detailing all the included tax reforms and revenue acts.

[15] See Weber (2014) for alternative exposition of this point.

Figure 1: Average Marginal Tax Rate Forecasting Errors and All Statutory Variation



Time series plot of U.S. Average Marginal Tax Rate one-step ahead forecasting errors (given by the larger BLP described below) in blue. All statutory variations in average marginal tax (IV1) in orange.

1. An endogenous reform is accidentally included in the dummy.
2. A shock arrives but is not included in the dummy
3. The statutory variation mismeasures the structural shock due sampling error.

The second source is unlikely to be important, as all exogenous changes in AMTR must do due either through tax reforms or revenue acts — and a comprehensive list of those changes are readily available. The third source, in contrast, is expected as both SV measures provided by Mertens and Montiel Olea (2018) only use a sample of the population. Additionally, tax calculators used to compute $\tau_t(income_{t-1})$ might be imprecise. Such measurement errors muddy identification if z_t were used directly but not in a two-stage regression.[16] The first source of discrepancy though violates the identification assumption. The instrument validity hinges upon no endogenous reform being included in the dummy.

A good AMTR instrument would be subjected only to the third possibility. In that case, the relation between shocks and instruments is given by:

$$z_t = \epsilon_t^q d_t + m_t \tag{54}$$

$$m_t \sim N(0, \sigma_m^2), \tag{55}$$

where m_t is measurement error. Since the first stage parameter is the projection parameter of $\text{Proj}(\epsilon_t^q | z_t)$. It is easy to see that:

[16] If they are assumed to be orthogonal to structural shocks and samples are large.

$$\pi = \frac{\sigma_q^2}{\sigma_q^2 + \sigma_m^2} \in [0,1]. \tag{56}$$

If the tax instrument is constructed correctly, the first stage parameter has to be less than one and it can be interpreted the proportion of the variance of the instrument explained by the structural shock. This provides a *necessary* condition to validate an instrument based on posterior distribution of π, as probability mass on regions $\pi > 1$ indicates an endogenous reform was included. That is the posterior distribution of the first-stage is identifying not just π but, in addition, the bias term in (12). Note that the sign of this bias term has to be positive, as SV is always positively correlated with z_t.

Figure 2: First Stage Parameter Distributions

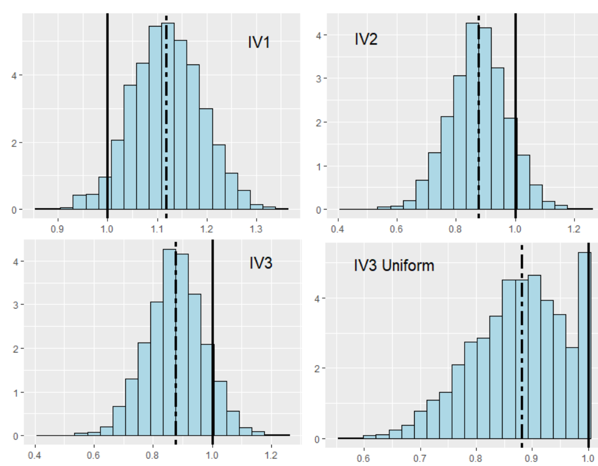


Figure 3: First stage parameter posterior distribution, obtained by running the first stage with each instrument separately, with all control variables (as described in the following section). Top left and right and bottom left all have the same disperse prior distributions for π, bottom right has $U(0,1)$ prior.

The graphs in Figure (2) show the posterior distribution of the first-stage parameter π for each of the three different instruments given the same dispersed uniform prior over $\pi \sim U(0,10)$ for all the cases, except for the bottom right where I use a $U(0,1)$ prior. According to the posterior distributions, the probability of $\pi > 1$ when z_t is the IV1 is 88%. The evidence points the first instrument is endogenous. Some narrative criteria to select the right reforms are needed. The posteriors for the remaining two instruments are close. This is expected as (56) is invariant wtih respect to the choice of reforms. While the stability of these posteriors is evidence the instruments are valid (otherwise endogenous reforms included in IV2 but not IV3 would shift the posterior towards 1), they do not amount to a sufficient condition, as the evidence is still consistent with bias together with a higher level of signal-to-noise ratio.

Regardless, this analysis is still informative of the quality of the instruments. The point estimates π for both IV2 and IV3 are approximately 0.83 and it is unlikely the sampling error on the construction of SVs measures could amount to more.

Alternatively, since the theoretical restriction is that $\pi \in [0,1]$, the more informative prior $U[0,1]$ could be used to eliminate the probability mass on the region $\pi > 1$. I advise against such practice, as the role of π in the econometric modeling is simply to give the second-stage estimates adequate scale. For this reason, as long as posterior π has enough mass far from zero, specific values of π shouldn't matter. That said, impulse responses for that case were estimated and provided in the online appendix.

4.1.3 Data and Model Specification

I run two BLP-IVs for each of the three instruments. The sample has an annual frequency and covers the period between 1948 to 2012.

I focus on two specifications. The first is a small four-variable system that includes the outcomes: aggregate taxable income, gross domestic product, effective federal funds rate, and unemployment rate. The policy measure is the AMTR time series. Controls include two lags of each outcome[17].

The second specification is medium-scale BLP-IV that includes all of the previous four outcomes as well as: household consumption, investment, and CPI inflation. The control variables include three lags of each outcome and additionally three lags of each: (1) federal government debt, real stock prices, government spending, Gertler and Karadi (2015) monetary policy surprise shocks, Ramey (2011) fiscal news shocks, Arezki et al. (2017) international oil supply news shock and a dummy with all NBER recessions. Posteriors distributions were simulated from 20,000 iterations of the Gibb's sampler algorithm described in the previous session.

4.1.4 Prior Distributions

Priors of the impulse responses β_h^k are assumed dispersed $N(0,100)$. The prior covariance matrix is the usual $IW^{-1}(\Sigma_0, 3)$ where Σ_0 is a diagonal matrix containing the variances of y_t^k, y_t^1. Priors over π, as described above, are $U(0,10)$. Finally, priors over the remaining auto-regressive, controls and moving average parameters are built assuming that each time series follows an AR $y = \mu_0 + 0.85Ly + e_t$ process, with μ_0 being the long-run mean of the respective time series. The tightness of each of those priors ($\alpha_0 = 0.89$) was chosen to reduce the root mean squared errors compared to classical LP-IV.

[17] All stock variables have been normalized so that IRs are in %.

4.1.5 Main Results

The panels of the left of Figures (4) and (6) showcase the impulse responses using IV2 and the right panels presents the results using IV1, using the medium-scale BLP-IV in both cases. Results for the model with IV3 were omitted as they don't differ substantially from the model with IV2. All figures display the median of the posterior IRF distribution and their 90% equal tail credible sets. My preferred estimate is given by the model with IV2.

Results from model with IV2 Across all seven outcomes, the effects in the same year of the reform implementation are small and are only statistically different from zero for GDP (-0.63%). This finding is consistent with the micro-level literature on marginal income tax changes who generally find small and non-significant effects on aggregate income. These results are usually attributed to either identification issues, difficulty in finding good model specifications, or myopic behavior by low and middle-income households, which may delay their reaction to changes in tax incentives only after they pay their taxes at least once. My results corroborate the last possibility.

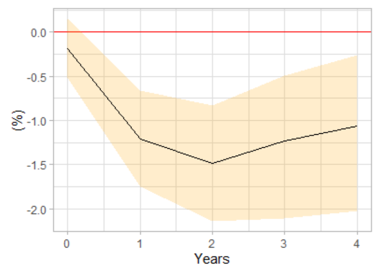
The bulk of the general equilibrium effects happens between the first and third years after the tax changes. Aggregate income drops by 1.25% in the first year, peaks at a 1.49% drop in the second year, and slowly revert back to the trend in the long run. The high persistence of AMTR shocks on aggregate income is expected since tax reforms permanently change household incentives. In contrast, the effects on GDP and Consumption peak in the first year (-1.21% and -0.75% respectively) and quickly return to their trends by the third year.

The effect on consumption is puzzling, as one would expect the long-run effect on taxable income to show up on consumption. The discrepancy between the impact on GDP and income is puzzling as well. Both can be attributed to a substitution effect that the BLP-IV is able to capture. Since U.S. income tax schedules incidence over wages is higher than over alternative sources of income, a raise in AMTR induces substitution of labor for those alternatives in the aggregate level. Such an effect is consistent with all estimates: the AMTR shock causes a large temporary increase in investment (2.11% in the first year and 3.76% in the second year) and a decrease in interest rates (-0.62% in the second year). As a result, economic activity and consumption recover faster than taxable income. This substitution effect is also identified in alternative specifications of BLP-IV using other labor market outcomes such as wages and labor force participation. [18]

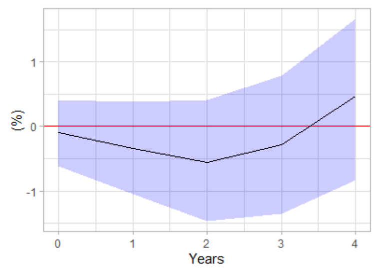
In contrast with all the other effects discussed above, the impact of AMTR shocks on the consumer price index (CPI) is highly persistent and only occurs in the long run. A 1% increase in AMTR causes

[18] Results for this version are available on request.

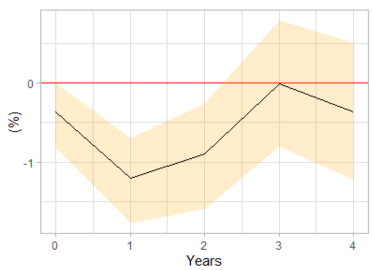
Figure 4: BLP-IV Responses to 1% AMTR shock.



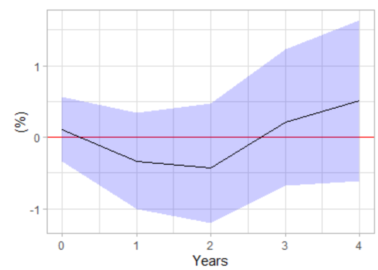
(a) Aggregate Income



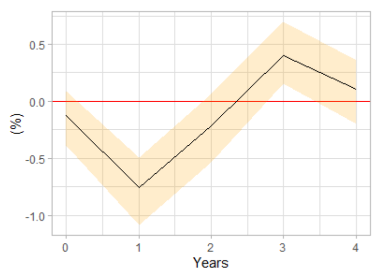
(b) Aggregate Income



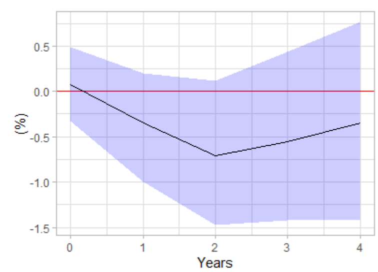
(c) GDP



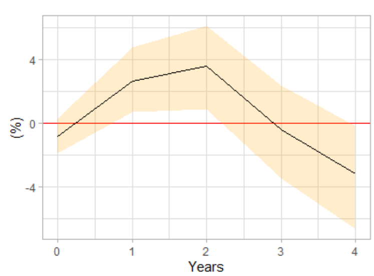
(d) GDP



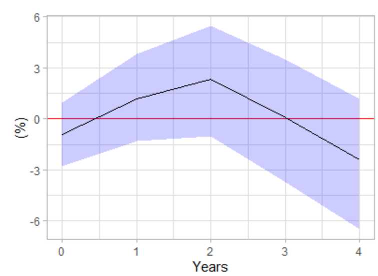
(e) Consumption



(f) Consumption



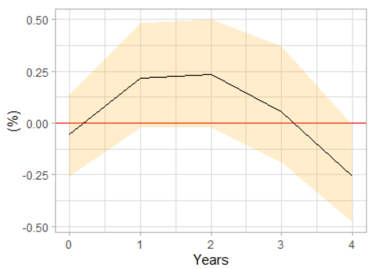
(g) Investment



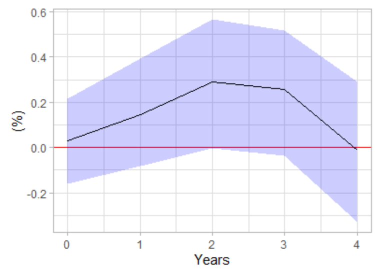
(h) Investment

Impulse Responses for 1% AMTR shock using all the control variables in the medium-size model. To the left side in orange, shocks are identified with IV2. To the right side in blue, shocks are identified with IV1. Black lines are the median of the posterior distributions and intervals are 90% equal-tailed credible sets.

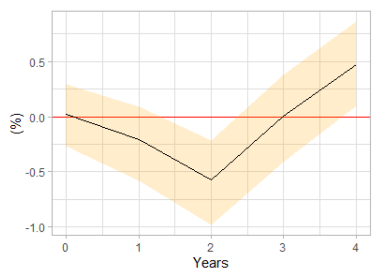
Figure 5: BLP-IV Responses: 1% shock to AMTR (continuation).



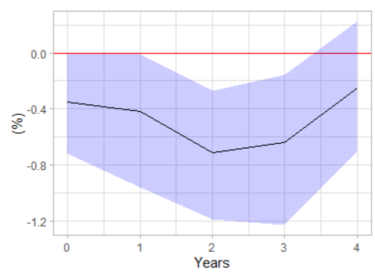
(a) Unemployment



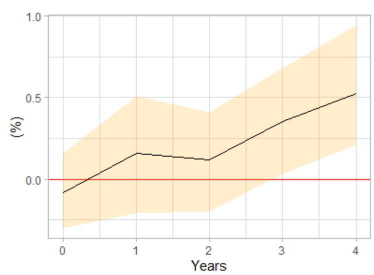
(b) Unemployment



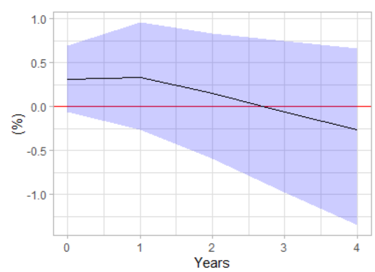
(c) Federal Funds Rate



(d) Federal Funds Rate



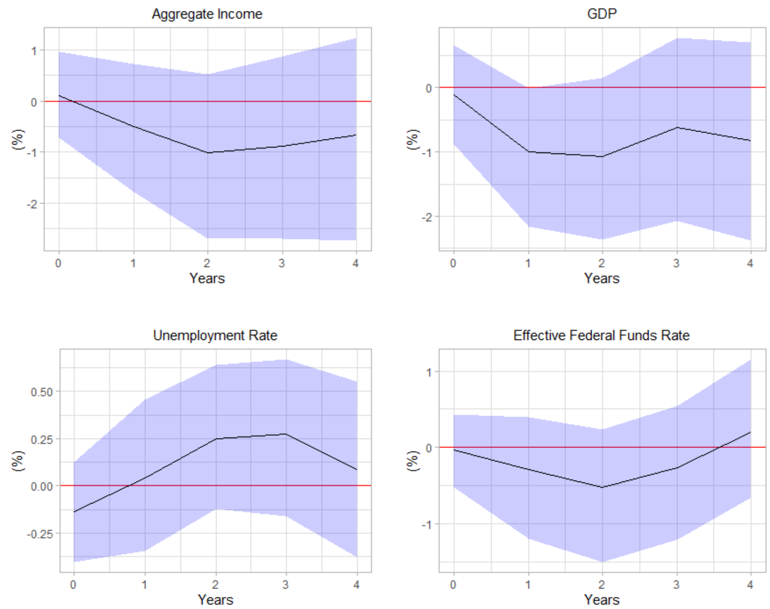
(e) CPI



(f) CPI

Figure 6: Impulse Responses for 1% AMTR shock using all the control variables in the medium-size model. To the left side in orange, shocks are identified with IV2. To the right side in blue, shocks are identified with IV1. Black lines are the median of the posterior distributions and intervals are 90% equal-tailed credible sets.

Figure 7: BLP-IV Responses to 1% AMTR shock (Small-Scale Model)



Impulse Responses for 1% AMTR shock using the small-scale model. In all cases impulse responses are identified with IV2. Black lines are the median of the posterior distributions and intervals are 90% equal-tailed credible sets.

0.5% inflation after four years. This finding is consistent with fiscal new-Keynesian DSGE models such as Bhattarai and Trzeciakiewicz (2017). A rise in labor income taxes reduces both aggregate supply, by raising the cost of labor input, and aggregate demand through the income effect channel. The effect on prices is ambiguous in the short run, but takes over in the long run because the income effect is temporary and the supply restriction is not.

4.1.6 Results from Model with IV1

In Section 4.1.2 I argued that the posterior density of π when estimated with the first IV1 implied that the dummy d_t had to include a few endogenous reforms. To further investigate whether that is the case, I estimate the model using IV1 to verify that the estimates are biased in the expected direction. Results are shown in Figures (

Endogenous tax reforms are expected to be pro-cyclical. For this reason, the AMTR instrument that includes all reforms may capture the effect of other shocks that positively impact economic activity, creating a bias in the opposite direction of estimates. The estimates for all seven outcomes confirm this intuition, although only in the case of CPI is the bias large enough to revese the direction of the impulse responses.

4.1.7 Comparison between Small-Scale and Medium-Scale Models

To showcase the merits of including a robust set of controls, I provide estimates of the small-scale model in Figure 7, utilizing IV2 and the same priors as the medium-scale models. The BLP-IV with dispersed priors over the impulse response is unable to identify even the first-order effects on aggregate income, when considering 1% AMTR shocks.

4.2 Monetary Policy Surprises

In the previous empirical application, the BLP-IV was used to provide estimates of impulse responses when samples are small and the instrument faces two possible threats to identification: violation of lag-exogeneity and possible inclusion of endogenous reform. In this section, I address a different kind of threat to identification in the form of weak instruments. To show this, I revisit Gertler and Karadi (2015) analysis of monetary policy surprises. The study attempts to identify the causal effects of monetary policy shocks on macroeconomic outcomes such as industrial production and inflation, by using Federal Open Market Committee (FOMC) unexpected interest rates announcement. Those instruments are constructed in a series of high-frequency event studies of the impact of surprise announcements on the settled price of public bonds at varying maturity and are available in Gertler and Karadi (2015) replication files.

This example was chosen because this is a widely used type of instrument[19] that is known to be weak[20] when two-year Treasury rates are used as policy rate. According to both Gertler and Karadi (2015) and, more generally the monetary policy literature, the one-year rate used in most empirical analyses does not adequately capture policy decisions that target the term structure. Also, there is tension in the literature about results. Using a Proxy SVAR, Gertler and Karadi (2015) finds an increase in one-year treasure rates causes a decrease in both CPI and industrial production. These results however are not replicated in classic LPs (Ramey (2016)). It remains an open question in the literature whether these differences are driven by non-invertibility bias in SVAR model or by LP less precise estimates (or both).

A third possibility is that the high-frequency instrument violates lag-exogeneity. If so, both models are biased then. This third possibility can be safely dismissed as it's one can to show the lag-exogeneity bias is exactly the same between SVARs and LPs. Additionally, evidence from medium-scale SVAR such as Caldara and Herbst (2019) is more or less consistent with taht of Gertler and Karadi (2015), suggesting not much bias reduction can be gained by using larger models.

[19]For recent publications, see Miranda-Agrippino and Ricco (2021) and Jarociński and Karadi (2020).

[20]The F-statistic varies between 5 and 8, depending on the policy rate used and the maturity of the instrument.

4.2.1 Data and Model Specification

To shed light on the issue, I estimate a BLP-IV using Gertler and Karadi (2015) specification and data, but with the two-year treasury bond rates as the policy rate. The sample has a monthly frequency and covers the period between 1988M10-2021M06.

The outcomes include log industrial production and log CPI. Controls include 12 lags of each outcome, the two-year Treasury rate, and excess bond premium. The instrument is the impact of FOMC surprise announcements on one-month maturity bond prices.

4.2.2 Prior Distributions

Disperse impulse responses prior distributions $N(0,100)$ are used to address weak-instrument concerns. To save on computational time, a classic LP-IV is estimated using a training sample that covers 1979M07-1988M10, and estimates are used as priors for all the remaining auto-regressive coefficients and the covariance matrix. Priors for the moving average coefficients are derived from the auto-regressive structure, as described in Section 32.

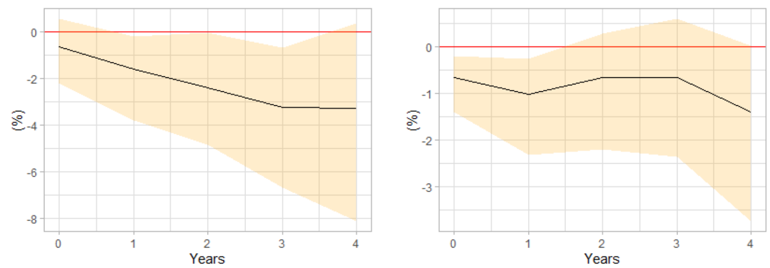
Priors for the first stage parameter is $\pi \in [0,100]$. This truncation to the positive region helps sharpen identification and such restriction reflects the theoretically-based assumption that Treasury rates should be positively correlated with bond prices.

4.2.3 Results

The impulse responses are presented in Figure 8, normalized to one standard deviation shock in the policy rate. The one standard deviation shock has a negative impact on both industrial production and CPI, although the impact on industrial production is not significant during the first six months. This impact is long-lasting and consistent with the body of evidence of monetary non-neutrality, even in the long run.

Qualitatively, the estimates are similar to those obtained in SVARs (both Bayesian and frequentist), unlike most local projections estimates that find a positive effect on activity (Ramey (2016)) or statistically insignificant effects (Lusompa (2021)). However, the magnitudes of the effects are *considerably* larger than most of the literature e.g. my estimated effect on industrial production is -3.32% after 48 months, while Gertler and Karadi (2015) find less than -0.5%, Ramey (2016) finds +0.5%, and Miranda-Agrippino and Ricco (2021) finds no long-run neutrality after four years. Since instruments are weak, one needs to be cautious when taking point estimates at face value. The interval estimates suggest the true effect after four years could be anywhere between 0% and -5% with 90% certainty. While these estimates are too

Figure 8: BLP-IV Responses to one standard deviation Monetary Policy Surprise shock



(a) Industrial Production (b) CPI

Impulse Responses for one standard deviation MP shock using. Black lines are the median of the posterior distributions and intervals are 90% equal-tailed credible sets.

imprecise to deliver a good quantitative measurement of monetary policy effect on economic activity, they show without ambiguity that monetary policy is counter-cyclical both in the short and in the long run.

Moreover, in many SVAR estimates that use high-frequency instruments, there is a ”year-one puzzle” — even if monetary policy is counter-cyclical in the long run, the effect of monetary policy on activity is positive during the first year. The evidence provided by the BLP-IV produces no such puzzle.

The results suggest that the discrepancy between classic SVAR and LP estimates cannot be attributed to invertibility issues in the SVAR and are driven by classic LPs lack of efficiency.

5 Concluding Remarks

I show a Local Projection parametric model can be derived when researchers have access to a sufficiently rich set of controls to propose a forecasting model for the outcome variables. In such a model, structural impulse responses can be identified through the usage of instrumental variables. A Gibb’s sampler that simulates posterior distributions of structural impulse responses is proposed and applied to estimate the dynamic, causal effects of average marginal tax rates (AMTR) shocks and monetary policy shocks.

The chief strength of the Bayesian framework, the incorporation of prior information in econometric modeling, is applied to address three key identification challenges. First, the often strong lag-exogeneity condition for instrument validity can be relaxed when LPs are defined over forecasting errors. Shrinkage priors over nuisance auto-regressive and moving average coefficients allow finite sample models to include a larger set of control, enhancing robustness.

Second, I show explicit assumptions about the joint distribution of the target-shock and the instrument have implications for the values of the coefficient of the first stage regression. In several cases, posterior inference of this parameter can be informative about the quality of the instruments and hence used by researchers as an additional robustness check. Additionally, prior information about this parameter, such as its sign, is often theoretically-justified and easily implemented, sharpening inference.

Finally, weak instrument concerns are addressed by the usage of dispersed priors over impulse responses. While this approach may potentially make estimates highly imprecise, however, the informative priors over the remaining parameters should be used to help macroeconomists reach qualitatively meaningful estimates. All these techniques were used in the two empirical applications to shed light on open empirical questions.

First, I identify the impulse responses of average marginal tax on medium-scale BLP, concluding marginal income tax shocks are counter-cyclical but their effect on economic activity lasts only two years, in spite of their lasting effect on taxable income. This is attributed to capital-labor displacement effects I'm able to identify. Additionally, I conduct validation of instruments commonly considered by the literature. Statutory variation of taxes without narrative selection of tax reforms is endogenous. The remaining tax instruments satisfy the necessary condition for exogeneity.

Second, I identify the impulse responses of monetary policy shocks as in Gertler and Karadi (2015), using two-year interest as the policy rate. Instruments are weak in that setting, but under additional prior assumptions, impulse responses are identified. Monetary policy shocks are counter-cyclical and their effect on output lasts until after 48 months.

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A Appendix 1 - Proof of Corollary 2

First, notice $e_{t+h,t-1}$ a is covariance stationary process. I will this assertion directly, by deriving its moments. To do this, I will introduce additional notation specific of this appendix. I also re-scale the shocks ϵ_t to one standard deviation, in order to simplify calculations.

Definition 3. *Let A be a $m \times n$ matrix. The term $\{A\}_\alpha$ refers to a linear combination of all columns of A, that is $\{A\}_\alpha \in span(A)$. The term α is a n-dimensional vector containing the scalars.*

From the representation (1):

$$e_{t+h,t-1} = \{\epsilon_{t:t+h}\}_\alpha - \beta_h^k \epsilon_t^q.$$

I proceed to prove its covariance stationarity by deriving its moments. First the mean:

$$E(e_{t+h,t-1}) = E(\{\epsilon_{t:t+h}\}_\alpha - \beta_h \epsilon_t^q)$$
$$E(e_{t+h,t-1}) = 0.$$

Now all the auto-correlation coefficients:

$$E(e_{t+h,t-1}e'_{t+h+s,t-1}) = E[(\{\epsilon_{t:t+h}\}_\alpha - \beta_h \epsilon_t^q)(\{\epsilon_{t:t+h}\}_\alpha - \beta_h \epsilon_t^q)']$$
$$E(e_{t+h,t-1}e'_{t+h+s,t-1}) = \begin{cases} \{I_{h+s}\}_{\alpha^2} + \beta_h\beta_h' \text{ for all } |s| = 0,1,2,...,h-1. \\ 0 \text{ otherwise} \end{cases}$$

Since $I_{h+s}\}_{\alpha^2} + \beta_h\beta_h' < \infty$ and it does not depends on t, $e_{t+h,t-1}$ is covariance stationary and admits Wold representation. That is, $\exists\ C(L) = \sum_{t=1}^{\infty} C_{-i}L^i$ such that $e_{t+h,t-1} = C(L)\zeta_t$ where ζ_t is white noise. Since $e_{t+h,t-1}$ auto-correlations are zero for $|s|$ equal or higher than h, the lag order of $c(L)$ is exactly h.

Does $e_{t+h,t-1}$ following a VMA(h) process imply that each individual projection error $e^k_{t+h,t-1}$ has MA(h) representation ? The answer is yes. To see this, write the projection errors as $e^k_{t+h,t-1} = re_{t+h,t-1}$ where r is vector of zeros with a single 1 on its k-ith entry. Since any continuous function of a covariance stationary process is also covariance stationary, $e^k_{t+h,t-1}$ admits its own univariate Wold representation and the lag order evidently cannot exceed h (or else $E(e_{t+h+s,t-1}e'_{t+h,t-1}) \neq 0$ for $s \geqslant h$).